

Two financing rounds, one company, one equilibrium

Elies Lucio

Hugues Pouget

Research showcase note 29 September 2026

Abstract

A venture's future financing opportunity changes the value of its present equity. This note constructs an equilibrium in which a privately informed manager raises capital immediately, preserves the company through its first payout, and raises fresh capital for a distinct second investment. The same company receives two positive settled primary financings with probability one. The construction includes every legal testing, bridge, delay, refusal and exit history, supported beliefs at those histories, and deviations that change a player's entire plan across both stages. Its equilibrium concept is full-history weak perfect Bayesian equilibrium. The proof below is self-contained for the specified institution.

Contents

1	The institution and its exact domain	2
2	Statement and the explicit policies	3
3	Accounting and receiving incentives	5
4	Supported beliefs and actual laws	6
5	Complete first-stage incentive comparisons	9
6	Joining the stages and proving the theorem	12
7	An exact witness and its economic interpretation	13

1 The institution and its exact domain

There is one persistent company identifier ι and five distinct actors: manager M , incumbent I , outsiders O_1, O_2 , and receiving financier F . Initially company treasury is c . The manager owns fraction a , has no personal injection account, and knows $\theta \in \{L, H\}$. The incumbent owns $b = 1 - a$, has verified cash B_0 , and privately observes $z \in \{-, +\}$. Outsiders have cash W_1, W_2 and no shares. Financier F has its own idle cash K from date zero, no first-stage role or shares, and no additional private economic signal. Everyone maximizes actual final personal cash; idle cash has gross return one and discounting is zero. There is no debt, wage, side claim or managerial private benefit.

The prior is $\mathbb{P}(H) = q$, with $\mathbb{P}(+ | H) = \eta$ and $\mathbb{P}(+ | L) = 1 - \eta$. A first indivisible input costs I_0 ; if bought by T , it yields actual Y_θ at T . Unbought inputs have no salvage. Before investment the manager may abandon, test, delay or file one formal equity round. Unused company cash is distributed under its actual old cap table on abandonment or uninvested termination.

Testing intensity lies in $[0, \bar{u}]$, is predictable and piecewise constant, and may be chosen initially and at at most J subsequent stopping-time interventions, with at most one intervention in a protocol phase. Mandatory public reports $+$ and $-$ have intensities $u\ell_\theta^+$ and $u\ell_\theta^-$. Testing burns company cash at rate κu and stops effectively at cash zero, without unsettled liabilities or a reset of the intervention budget. At ordered bridge dates, I may make no offer or offer cash/equity terms

$$0 \leq k \leq \min\{B, (I_0 - C)_+\}, \quad 0 \leq d \leq 1.$$

Here C, B are current company and incumbent cash. Acceptance transfers k and replaces every old fraction s_j by $(1 - d)s_j + d\mathbf{1}_{\{j=I\}}$. Rejection changes nothing. Zero-cash equity gifts are legal.

A formal round with $C < I_0$ seeks $R = I_0 - C$ for total new fraction $\alpha \in [0, 1]$. Simultaneously, I reserves $k \in [0, \min\{B, R\}]$ and outsider i reserves either zero or R , backed by its own money. With $m \geq 1$ underwriting outsiders, actual allocations are $x_I = k$ and $x_i = (R - k)/m$ for each underwriter. With none, only $k = R$ funds. After allocations become public, M consents or refuses. Funded consent transfers only the allocations, refunds unused reserves, issues

$$s'_j = (1 - \alpha)s_j + \frac{\alpha x_j}{R}, \quad (1)$$

and spends $C + R = I_0$ on the input. Failure or refusal refunds all reserves, issues nothing and liquidates C under the pre-issue table. No retry is available. The auxiliary rule at $C \geq I_0$ permits private purchase for I_0 with residual $C - I_0$; the invariant proved below makes this rule unreachable from the domain of this theorem.

Investment ends testing and bridges. At a bridge date the order is any report first, incumbent offer, manager response, then remaining legal control/filing choices. At other dates reports precede interventions. Filing begins an atomic proposal–commitment–allocation–consent–settlement/refund sequence, with no waiting or additional control inside. At T , the last permitted filing decision and its entire atom precede the output/payout consequence.

A completed first-stage boundary is *live* exactly if the first input has been bought and the company has not liquidated. All other completed original boundaries are absorbing exits. At T , Y_θ and any residual treasury are paid once under the first table; ordinary shares survive this dividend. A live company enters the receiving round at

$$t_2 = T + \Delta, \quad t_3 = T + 2\Delta.$$

Treasury is then zero. The manager abandons or posts $\lambda \in [0, 1]$. Financier F reserves zero or its actual K ; after K , the manager consents or refuses. Funded consent transfers K , buys a distinct input costing K , and sets fractions to $((1 - \lambda)s, \lambda)$ for the old owners and F . This input pays the public deterministic Z at t_3 . Failure, refusal or abandonment returns any reserve once and absorbs with zero company treasury. Receiving has no testing, bridge, waiting, loan, reinvestment of old personal dividends or retry. The manager retains authority at zero equity; passive owners have singleton receiving actions. No opportunity remains after t_3 .

Exact primitive domain. All coordinates are finite, and precisely the following numerical restrictions apply:

$$\begin{aligned}
I_0 > 0, \quad Y_H > Y_L > I_0, \quad 0 \leq c < I_0, \quad 0 < B_0 < I_0 - c, \\
0 < a, b < 1, \quad a + b = 1, \quad 0 < q < 1, \quad \frac{1}{2} < \eta < 1, \\
T, \kappa, \bar{u} > 0, \quad J \in \mathbb{Z}, \quad 1 \leq J < \infty, \quad N \in \mathbb{Z}, \quad 2 \leq N < \infty, \\
0 < t_{br,1} < \dots < t_{br,N} < T, \quad 0 < \ell_\theta^\epsilon < \infty \quad (\theta = L, H; \epsilon = +, -), \\
I_0 \leq W_1, W_2 < \infty, \quad 0 < K < Z < \infty, \quad 0 < \Delta < \infty.
\end{aligned} \tag{2}$$

The structural and information rules just described are also part of the domain. In particular $c = 0$ and $W_i = I_0$ are included.

Full histories and admissibility. Public records retain all actual times, actions, reports, allocations, free/reserved cash, historical and current ownership, company identity, input and payment flags, remaining opportunities and phase. Each actor also keeps its actual private type/signal and complete private randomization/memory word, at its actual availability times. These spaces are standard Borel spaces, represented by countable unions of finite tagged chronological words, together with the supplied standard Borel private records. All complete Borel nonanticipative policies, legal stopping times and permitted correlated own randomization remain admissible. A private randomizer is not an additional hard observation of another actor's economic type. Fixed public company/fund dates and profile marks are nonsignalling; any permitted bounded Borel annotation is read only at its declared existing information time. The selected source uses empty randomization records; feasible nonempty records, including F 's, remain in the history and deviation domains.

2 Statement and the explicit policies

A *full-history weak perfect Bayesian assessment* here consists of information-feasible Borel policies, one generated chronological law, and public/whole-own kernels on complete compatible books. On the generated law the kernels are conditional-probability versions. At null readouts they preserve the actual own record and hard physical/accounting/output support. Identical current readouts have identical kernels, including across chronological handoffs; selected continuations satisfy Bayes/development identities whenever the relevant normalizer is positive. At each feasible own history, the selected remaining plan maximizes expected final cash over every complete admissible unilateral remaining plan, evaluated from that history's kernel with opponents fixed as functions of their changed actual observations. This weak concept imposes neither a common tremble sequence nor arbitrary-null tower identities.

Theorem 1. *For every primitive vector satisfying (1) and the stated institution, the policies and kernels constructed below form a full-history weak perfect Bayesian equilibrium. At every*

live first-stage boundary its selected continuation funds the receiving input; every original exit remains absorbing. Under its single unconditional equilibrium law, the same company receives positive settled primary capital $R_0 = I_0 - c$ at date 0 and K at date $T + \Delta$, with probability one. Actual first output is Y_θ at T and actual second output is Z at $T + 2\Delta$, distributed under their respective cap tables.

Set

$$S = Z - K, \quad V_\theta = Y_\theta + S, \quad R_0 = I_0 - c, \quad r = K/Z.$$

V_θ is a terminal incentive abbreviation, never current spendable cash or the actual first output. Define

$$\begin{aligned} \pi_0(H, +) &= q\eta, & \pi_0(H, -) &= q(1 - \eta), \\ \pi_0(L, +) &= (1 - q)(1 - \eta), & \pi_0(L, -) &= (1 - q)\eta, \end{aligned}$$

$\rho_z = \sum_\theta \pi_0(\theta, z)$ and $\mu_z = \sum_\theta \pi_0(\theta, z)V_\theta/\rho_z$. Put

$$P = \frac{(R_0 - B_0)\rho_+ \mu_+ + R_0 \rho_- \mu_-}{R_0 - B_0 \rho_+}, \quad \alpha_p = R_0/P. \quad (3)$$

An untouched initial prefix is INITIAL. Any initial departure, and every other pre-round prefix, is permanently LOW; equality of numerical date or cash cannot restore the initial word.

At the initial free choice, M files immediately with α_p . At that selected proposal I contributes $k_z = B_0 \mathbf{1}_{\{z=+\}}$, and both outsiders reserve R_0 . At every LOW free filing choice, M files immediately with

$$Q(C) = V_L - I_0 + C, \quad \alpha_L = (I_0 - C)/V_L, \quad h_L = 1 - C/V_L.$$

If a control announcement is currently available it chooses zero first; if controls are exhausted or a control was already chosen it files directly. Mandatory report/bridge phases are respected. Incumbent I makes no offer at every bridge opportunity. At an actually pending offer (k, d) , M accepts exactly when

$$d \leq \frac{k}{Q(C) + k}, \quad (4)$$

then follows the actual phase order and immediately files at the resulting books. At every funded first round the selected consent rule is

$$\mathbf{1}_{\{(1-\alpha)V_\theta \geq C\}}, \quad (5)$$

including at zero manager stake. At each LOW proposal, including a nonpool initial proposal, the simultaneous response is

Posted fraction	Incumbent contribution	Each outsider reserve
$\alpha < \alpha_L$	0	0
$\alpha = \alpha_L$	0	R
$\alpha_L < \alpha \leq h_L$	B	R
$\alpha > h_L$	0	0

At every live receiving entry M posts r ; F commits K iff $\lambda \geq r$; M consents after every actual K commitment, even below r and at zero gain. Exits and passive phases use their singleton continuation. Earlier sunk actions are never undone by these prescriptions.

3 Accounting and receiving incentives

Lemma 1 (Reachable cash and terminal receipts). *At every noninvested descendant of the primitive domain, $0 \leq C < I_0$ and $0 \leq B < R = I_0 - C$, and each outsider can afford R . Before the formal round only M, I own shares. If a first round settles with old fractions s , allocations x and current personal cash c_j , its final cash under selected receiving follow-through is*

$$c_j - x_j + ((1 - \alpha)s_j + \alpha x_j/R) V_\theta \quad (6)$$

for an old actor. Failure/refusal instead gives $c_j + s_j C$. Financier F finishes with K .

Proof. If cumulative accepted bridge capital is K_{br} and evidence expense is $E \geq 0$, then

$$C = c + K_{br} - E, \quad B = B_0 - K_{br}, \quad R - B = I_0 - c - B_0 + E > 0.$$

Actual accounts are nonnegative, and $W_i \geq I_0 \geq R$. Bridge dilution does not give outsiders shares. With $B < R$, at least one outsider is necessary and sufficient to fund. Allocations sum to R and (1) sums to one. Settlement spends I_0 and leaves no residual treasury. The first table obtains Y_θ at T and, when second fraction r is issued, obtains $(1 - r)Z = S$ later. This proves (6). On failure/refusal all reserves return and only C is liquidated. At receiving, F pays K then receives $rZ = K$, so its original cash is not counted twice. $\square$

For a completed first boundary let B_j be actual retained personal cash, s_j its first fraction, and P_{old} indicate that the first dividend is unpaid. With actual residual treasury C_{res} , the selected boundary cash functional is

$$\hat{V}_j = \begin{cases} B_j + P_{old}s_j(C_{res} + Y_\theta) + s_j S, & \text{live,} \\ B_j, & \text{exited,} \end{cases} \quad \hat{V}_F = K. \quad (7)$$

Already paid dividends are inside B_j . On reachable formal-round boundaries $C_{res} = 0$; the displayed formula also traces the separately defined auxiliary private-purchase rule.

Lemma 2 (Receiving envelope, at every phase). *From any live completed first boundary, every complete unilateral receiving plan yields its deviator at most (7), with equality for the selected receiving suffix. At every already begun receiving phase the selected remaining suffix is also optimal, with that phase's actual sunk actions retained.*

Proof. Develop to entry by paying only the pending first dividend. Write b_j for the resulting personal cash. A funded consenting second issue gives an old owner $b_j + (1 - \lambda)s_j Z$ and F final cash $K + (\lambda Z - K) = \lambda Z$. Nonfunding gives b_j, K respectively. Reserve and refund are changes of account location until a payment actually settles.

For a deviating manager, fixed F declines $\lambda < r$. At $\lambda \geq r$, any consent yields increment $(1 - \lambda)s_M Z \leq s_M S$; refusal and abandonment yield zero. Posting r and consenting attains $s_M S$, including $s_M = 0$. For deviating F , fixed M posts r and consents; decline and commitment both give K . Passive owners have no receiving choice and receive $b_j + s_j S$. Exits have no receiving action.

At an already posted λ , F compares K with λZ ; their difference is $Z(\lambda - r)$, establishing its threshold including equality. The manager cannot replace its sunk proposal. If K is actually committed, consent has gain $(1 - \lambda)s_M Z \geq 0$, at every $\lambda \in [0, 1]$, including $\lambda = 0, 1$ and $s_M = 0$. After commitment F cannot withdraw. After recorded consent, input purchase, refusal or failure only mandatory transactions remain. Thus the exact selected phase values are

Current phase	Old owner j	Financier F
Entry	$b_j + s_j S$	K
Posted λ	$b_j + \chi(1 - \lambda)s_j Z$	$K + \chi(\lambda Z - K)$
Actual K ; before consent	$b_j + (1 - \lambda)s_j Z$	λZ
Consent/input; payout unpaid	$b_j + (1 - \lambda)s_j Z$	λZ
Abandonment/failure/refusal	b_j	K
Payout already completed	Actual current cash	Actual current cash

Here $\chi = \mathbf{1}_{\{\lambda \geq r\}}$. At a finer bookkeeping cut, value equals free plus reserved cash minus remaining mandatory outward payments plus remaining mandatory receipts. This handles refunds and outputs once only. In particular an entry value cannot be reused after a sunk donation or overpriced commitment. Every comparison is bookwise, so integrating it against any compatible actual joint distribution, including one created by earlier deviations, preserves it. $\square$

4 Supported beliefs and actual laws

We provide both a law for every complete comparison and one explicit everywhere belief system. This also makes clear which beliefs are used when a deviation changes the histories reached.

4.1 A chronological law for complete policies

Lemma 3. *For any supported starting kernel and any admissible complete profile, including a complete unilateral deviation, the described game generates a Borel chronological continuation law, retaining all actual private records and their correlations. Final cash is bounded for each fixed primitive vector.*

Proof. A probability kernel into a standard Borel action space can be realized by a uniform random variable: embed the space Borel-isomorphically in a Borel subset of $[0, 1]$ and take the generalized inverse of its distribution function. Its threshold sets are countable combinations of rational distribution-function inequalities, so the realization is Borel; null exceptional values can be assigned a fixed admissible output. A simulator coordinate is not a new observation: expose only the action and memory update prescribed by the original policy. Existing private sources are retained jointly, and a player's correlated full plan is not redrawn at successive decisions. Countably many specified noise coordinates have the probability measure defined by their consistent finite successive-kernel integrals.

Use an independent Poisson candidate clock of rate

$$L = \bar{u} \max_{\theta} (\ell_{\theta}^{+} + \ell_{\theta}^{-}) < \infty.$$

It can be constructed by exponential gaps; its count on $[0, T]$ has Poisson distribution with finite mean LT . At a tick, a fresh uniform selects $+$ with probability $u_{\text{eff}} \ell_{\theta}^{+} / L$, $-$ with probability $u_{\text{eff}} \ell_{\theta}^{-} / L$, and otherwise no report. Rates are the predictable pre-report rates. A rejected tick is invisible and redraws no private plan. Independent increments and these thinning probabilities give, first for elementary predictable integrands and then by monotone approximation, the required intensities $u_{\text{eff}} \ell_{\theta}^{\epsilon}$.

To execute arbitrary legal stopping times, take the quiet continuation from a resolved prefix up to the next report or mandatory boundary. The policy's next stopping time on that

continuation is a Borel function: rational stopping-time sublevel sets and the Borel quiet-path insertion prove this. Until the next new observation the true and quiet own histories coincide, so their stopping decisions coincide; otherwise a rational date strictly between two different decisions contradicts nonanticipation. At a report, append it first and reevaluate. This permits atoms, singular randomized stopping times and immediate report-triggered stopping.

Between events cash follows $C(s) = C - \kappa u(s - t)$ until its zero cutoff (or remains fixed if $u = 0$). Advance to the earliest elective stop, mandatory date, cutoff, candidate tick or T , and apply the actual phase rule. There are at most $J + 1$ control announcements, N bridges, $N + 1$ positive-cash depletion episodes, one first formal atom and finitely many receiving steps. Actual reports are bounded by the finite candidate count. Therefore this chronological recursion terminates almost surely. Every finite step is Borel; partitioning by its finite termination count gives a Borel complete-path map. Its pushforward of the joint initial kernel and specified noise measure is the asserted law. The same induction gives identical prefix laws for profiles that agree up to a completed boundary. It requires no equilibrium premise or topology on the space of all policies.

Accounts and claims are nonnegative; at most one input and one output of each kind occur. In particular all legal final cash satisfies

$$C_M^{\text{fin}} \leq Y_H + Z, \quad C_I^{\text{fin}} \leq B_0 + Y_H + Z, \quad C_{O_i}^{\text{fin}} \leq W_i + Y_H + Z, \quad C_F^{\text{fin}} \leq Z.$$

Thus every stated fixed-vector payoff integral is finite. All feasibility and Borel claims concern the actual raw history versions of policies, including their legal null histories. $\square$

4.2 Explicit full-book conditional kernels

Set $\sigma_H(+) = \eta$, $\sigma_L(+) = 1 - \eta$, and $\sigma_\theta(-) = 1 - \sigma_\theta(+)$. Classify each retained public word before any actual Y :

- A : untouched initial pre-proposal, or its selected α_p post before commitments are revealed.
- B_z : that prefix followed by exactly $(k, e_1, e_2) = (B_0 \mathbf{1}_{\{z=+\}}, R_0, R_0)$ and, if observed, yes consent; successful pre-output descendants keep the class.
- L : every other pre-output readout, including unexpected commitments and original refusal/failure/exit words.

These are observable prefix predicates. Their hidden-type weights are as follows; θ_j, z_j denote the conditioning actor's actual own type/signal.

Class	Public, O_1, O_2, F	Manager M	Incumbent I
A	π_0	$\delta_{\theta_j} \otimes \sigma_{\theta_j}$	$\pi_0(\theta \mid z_j) \otimes \delta_{z_j}$
B_z	$\pi_0(\theta \mid z) \otimes \delta_z$	$\delta_{\theta_j} \otimes \delta_z$	If $z_j = z$: $\pi_0(\theta \mid z_j) \otimes \delta_{z_j}$; otherwise $\delta_L \otimes \delta_{z_j}$
L	$\delta_L \otimes \sigma_L$	$\delta_{\theta_j} \otimes \sigma_{\theta_j}$	$\delta_L \otimes \delta_{z_j}$

After actual Y_ϑ , impose $\theta = \vartheta$. If the successful first atom was B_z , public/outside/F/M use $\delta_\vartheta \otimes \delta_z$; otherwise they use $\delta_\vartheta \otimes \sigma_\vartheta$. The incumbent always uses $\delta_\vartheta \otimes \delta_{z_j}$. The manager's feasible record necessarily has $\theta_j = \vartheta$. These weights persist through receiving. An original exit without output retains its pre-output L row.

To turn weights into full-book kernels, define $L_{\text{pub}}(h, \theta, z)$ by copying the entire public word and ledger/ownership fields, inserting compatible θ, z , and choosing empty unused randomization in unobserved private slots, with their mechanically determined observation and availability

records. Define $L_j(u_j, \theta, z)$ identically, except that actor j 's entire actual own private word is copied verbatim and its true own type/signal is imposed. Only compatible positive-weight pairs are used. Set

$$\begin{aligned}\beta_{\text{pub}}(G \mid h) &= \sum_{\theta, z} w_{\text{pub}}(\theta, z \mid h) \mathbf{1}_G(L_{\text{pub}}(h, \theta, z)), \\ \beta_j(G \mid u_j) &= \sum_{\theta, z} w_j(\theta, z \mid u_j) \mathbf{1}_G(L_j(u_j, \theta, z)).\end{aligned}\tag{8}$$

Lemma 4. *These are Borel supported probability kernels, preserve every whole-own record, and satisfy the weak consistency requirements. Under the selected source and policies they are conditional versions of one actual law.*

Proof. First verify the range of the constructors. Type and signal change no legal action menu, which is determined by copied phase/resources. A finite legal public action word can be realized by deterministic past policies along that word. Exact real action values do not require randomization. Strict positivity of all diagnostic intensities gives the same physical support before output for both types: a report under zero effective intensity remains impossible, while a legal positive-intensity report word introduces no new type-specific zero. Both signals have positive probability under either type. Actual Y imposes its matching type and is never inserted on an exit. Thus the copied books are legal and have the required public/own readout. Choosing unobserved private words in a null kernel changes no known record and does not replace a supplied joint deviation source. Coordinate copying and finite type insertion are Borel, so (8) is Borel. No measurable-section assertion is inferred merely from nonempty fibres.

At A , the selected post has likelihood one. The selected commitment vector has normalizers ρ_z publicly and for outsiders, $\sigma_{\theta_j}(z)$ for M , and one for the incumbent's own selected contribution (zero for the opposite selected-looking vector). All positive updates are exactly the B_z row. Allocation, consent and settlement add no information: both types consent, as proved below. At L , the selected commitment table pools in z ; its vector has likelihood one, and every alternative has zero selected likelihood. Consent depends only on actual θ , so every positive consent/refusal update preserves the corresponding row. Selected noninitial continuations file immediately and introduce no positive testing interval; there is no omitted selected report likelihood. Arbitrary past report words receive the supported L row directly.

At unpaid B_z books both output normalizers are positive for public/outside/ F and a matching incumbent; M 's actual output has probability one. A wrong- z incumbent and public/investor L kernels give masses $(1, 0)$ to (Y_L, Y_H) . At a positive normalizer, restrict the finite sum to the observed output and develop the actual ledger; this gives the displayed post-output row. At zero Y_H normalizer, the post-output H row is explicitly supplied without division. A true- H manager's positive own update agrees with that current own row even when the public output normalizer is zero.

For any deterministic chronological append D without new private innovation, coordinate copying gives

$$D(L_j(u, \theta, z)) = L_j(D_j u, \theta, z)$$

on its compatible image, with actual output restriction where appropriate. Paying the old dividend clears its pending flag once; development preserves surviving shares and every private record. Receiving actions are type-independent Borel maps; their selected likelihoods are one on compatible books. Unselected legal receiving words use the same copied-book rule. Original exits have only deterministic absorption. These identities prove positive-normalizer development and overlap: identical current readouts use identical predicates, weights and constructors, independently of a hidden earlier handoff label. A newly supplied legal private

innovation is copied at its actual availability time; it is not falsely described as the image of a deterministic append that draws none.

Finally let $\omega_{\theta,z}$ be the path generated by the selected pure policies from types (θ, z) and empty randomization records. Lemma 3 supplies its actual chronology. Its law is

$$\mathbb{P}^* = \sum_{\theta,z} \pi_0(\theta, z) \delta_{\omega_{\theta,z}}. \quad (9)$$

On these four paths, commitments reveal z because $B_0 > 0$, and actual output reveals θ . Grouping these four atoms by equal public or whole-own readouts proves the conditional-probability identity for (8). The copied on-law books equal the actual books since all selected randomization records are empty. At null own records their whole word is instead retained by construction. This proves the asserted common-law matching and supported completion, without imposing a tremble refinement. $\square$

5 Complete first-stage incentive comparisons

All comparisons in this section use the terminal cash (6), equivalently (7), with the selected receiving suffix. They are evaluated from the relevant *starting* whole-own kernel. A later off-path observation under a deviation does not replace that experiment's posterior by an assessment-null belief.

5.1 Initial price and the full formal round

All ρ_z are positive. Direct subtraction gives

$$\mu_+ - \mu_- = (V_H - V_L) \frac{q(1-q)(2\eta-1)}{\rho_+ \rho_-} > 0.$$

Both posterior types have positive mass, so $V_L < \mu_- < \mu_+ < V_H$. The denominator in (3) equals $(R_0 - B_0)\rho_+ + R_0\rho_- > 0$. Both weights are strictly positive, hence

$$V_L < \mu_- < P < \mu_+ < V_H, \quad 0 < \alpha_p < \alpha_L(c) < h_L(c) \leq 1. \quad (10)$$

In particular $(1 - \alpha_p)V_\theta \geq V_L - R_0 > c$, and likewise $(1 - \alpha_L)V_\theta > C$. Both types therefore actually consent at the selected terms. The price also solves the outsiders' residual-capital break-even equation; it was not obtained by averaging valuations while ignoring signal-dependent allocations.

At any funded history, manager consent gain is $a_h[(1 - \alpha)V_\theta - C]$, where a_h is its current old fraction. Rule (5) maximizes its two actions. At $a_h = 0$ it is an optimal *undivided* tie selection; this keeps the investor response table valid.

At simultaneous subscription, the fixed outsiders cannot see a forced incumbent choice k . Since $B < R$, the funding flag depends only on whether an outsider commits. Consent (5) is also independent of k . If f is the accepted-funding flag, its pointwise value is thus the same for all $k \in [0, B]$, and the full incumbent objective is

$$B + b_h C + \mathbb{E}\{f b_h [(1 - \alpha)V_\theta - C]\} + k \mathbb{E}\{f(\alpha V_\theta / R - 1)\}. \quad (11)$$

This retains legacy equity and all correlations; no independence factorization is used.

At the initial pool $f = 1$. The incumbent's objective given actual z is

$$B_0 + b(1 - \alpha_p)\mu_z + k(\mu_z/P - 1).$$

By (10), the global maximizer on the *entire* interval is B_0 at $+$ and zero at $-$. An outsider's commit-minus-abstain gain, given the other underwrites, is

$$\sum_z \frac{\rho_z(R_0 - k_z)(\mu_z - P)}{2P} = 0, \quad (12)$$

by (3). The actual allocation $(R_0 - k_z)/2$, not the gross reserve, is paid. Declining also yields zero; selected underwriting is optimal. These conditional pure-action inequalities integrate over all permitted mixtures and remembered own randomization.

At every LOW or nonpool initial proposal, investors' starting kernels put probability one on L . The four response rows are globally optimal:

1. If $\alpha < \alpha_L$, all incumbent contributions fail with the outsiders fixed at zero. A sole outsider deviation funds but gives $\alpha V_L - R < 0$, whereas declining gives zero; low quality consents in this region.
2. If $\alpha = \alpha_L$, both outsiders fund and consent is strict. Incumbent slope in (11) is zero, so zero is optimal; either outside action gives zero gain.
3. If $\alpha_L < \alpha \leq h_L$, low quality consents, including the tie at h_L . Incumbent slope $d_L = \alpha V_L/R - 1 > 0$ gives maximizing choice B . Each outsider obtains strictly positive gain $(R - B)d_L/2$ from underwriting.
4. If $\alpha > h_L$, fixed outsiders decline and every incumbent contribution fails. A sole outsider deviation is refunded because the believed low manager refuses, so abstention is an optimal tie.

Unexpected joint commitments are later observations, not an opportunity to revise simultaneous actions. There only the manager acts, and its consent comparison already covers both actual types.

For a free manager proposal, let $\alpha^* = \alpha_p$ initially and α_L otherwise. At α^* funding and consent yield $a_h(1 - \alpha^*)V_\theta \geq a_h C$. Every smaller alternative receives no outside funding; every larger alternative yields at most the selected payoff whether it settles, fails or is refused. The same proposal with contingent refusal cannot improve. This is a bound on every realized proposal and subsequent consent branch, so it covers continuum-valued mixed proposals and correlated complete consent plans. At zero stake all choices tie. Already posted terms are sunk; the remaining selected consent is optimal at those actual terms, with no free refiling granted.

5.2 Testing, timing, bridges and all future choices

Define

$$D_0 = V_L - I_0 > 0, \quad \beta_\theta = V_\theta/V_L \geq 1, \quad F_\theta(a_h, C) = a_h\beta_\theta(D_0 + C).$$

The liquidation gap is

$$F_\theta(a_h, C) - a_h C = a_h\{\beta_\theta D_0 + (\beta_\theta - 1)C\} \geq 0. \quad (13)$$

Consider an arbitrary complete adaptive manager plan from a noninitial free filing choice. Fixed incumbent opponents make no future bridge offers. Until purchase or liquidation its fraction remains a_h and cash is $C_t = C - E_t \leq C$. If the input is never bought, its company

receipts are $a_h C_\tau \leq F_\theta(a_h, C)$. If a formal round settles, the fixed LOW responses require $\alpha \geq (I_0 - C_\tau)/V_L$; thus

$$a_h(1 - \alpha)V_\theta \leq F_\theta(a_h, C_\tau) = F_\theta(a_h, C) - a_h\beta_\theta E_\tau \leq F_\theta(a_h, C). \quad (14)$$

These are pathwise bounds regardless of reports, intensity changes, stopping, randomized proposals and contingent consents. Immediate α_L filing attains the bound. Actual already retained personal cash can be added to each side once.

This attainment respects every phase. At a mandatory bridge date, process the no-offer before filing. At a report, append it before the next legal action. Set testing to zero only if a control action remains available; otherwise file directly. Filing consumes no positive duration and investment stops inherited testing. Exhausted controls are never reset. At T the last filing atom executes before output, and the same value is attained at T itself, rather than merely as a limit.

At an actual pending bridge offer (k, d) , both acceptance and rejection lead to the domain of (14). Their optimized manager-value difference is

$$F_\theta((1 - d)a_h, C + k) - F_\theta(a_h, C) = a_h\beta_\theta[k - d(Q(C) + k)]. \quad (15)$$

The denominator $Q(C) + k > 0$ makes (4) optimal, including equality and $a_h = 0$. This optimizes the response *and all subsequent actions* on both branches.

Now consider any complete incumbent offer-plus-later-subscription plan from a bridge history. Its starting kernel fixes $\theta = L$. With no offer, fixed M immediately files α_L and its terminal cash is

$$B - x + [b_h(1 - R/V_L) + x/V_L]V_L = B + b_hQ(C)$$

for every later contribution $x \in [0, B]$. If a bridge is accepted, any later legal subscription likewise yields

$$B - k + (b_h + a_h d)(Q(C) + k).$$

Its gain over no offer is

$$a_h[d(Q(C) + k) - k] \leq 0 \quad (16)$$

by the manager's acceptance rule. A rejected offer gains zero. Fixed M files immediately after the response, before any distinct later bridge date, so no repeated-offer plan escapes this bound. At an earlier open interval it files before the next bridge; at a sunk offer only the actual remaining subscription is compared. All continuous offers, gifts and correlated randomizations are covered by the branchwise bound.

At the initial manager choice, a nonpool immediate proposal, abandonment or any initial continuation enters the LOW bound (or liquidation). The selected pool exceeds that bound by

$$a(1 - R_0/P)V_\theta - F_\theta(a, c) = aV_\theta R_0(1/V_L - 1/P) > 0. \quad (17)$$

Keeping the selected proposal while changing later consent cannot improve it. These exhaustive first-action cases prove optimality of the complete initial plan, not only its first action.

An outsider has one first-stage underwriting decision and no later first-stage action. From a precursor, fixed manager timing leads to the already verified initial or LOW round; any incumbent bridge phase uses the actual resulting books. Its binary or randomized whole-own plan therefore obeys the proved conditional inequality, integrated under that same precursor experiment. The receiving financier has no first-stage move. Once an input has been purchased, or an exit recorded, all first-stage moves are terminated; passive comparisons are equalities. At a sunk proposal or contribution, only the genuine remaining phases are compared. This covers every feasible first-stage own start.

Boundary cases. No proof divides by an equity fraction or B . At $B = 0$ the incumbent menu is a singleton. At $C = 0$, $h_L = 1$ and $\alpha = 1$ uses the specified equality-consent row. At zero manager equity, authority and the undivided consent/bridge ties remain. A donation at $\alpha = 0$ is legal and genuinely costs its donor. Full bridge gifts remain feasible histories even if not selected. The invariant in Lemma 1 prevents auxiliary $C \geq I_0$ or incumbent-only closure from being reached; it does not remove their separately defined accounting rules.

6 Joining the stages and proving the theorem

Fix any feasible whole-own starting record u_j and any complete two-stage unilateral plan δ_j . Start from its supplied kernel $\beta_j(\cdot | u_j)$ and use Lemma 3 with fixed opponent policies at the actual observations produced. Suppose first that the start precedes completion of stage one, possibly inside a formal atom. Restrict the *entire* plan to its remaining first-stage prescriptions, retaining the same information, correlated source, future legal first-stage innovations and all memory. Replace only its receiving suffix by the selected suffix; call this the hybrid plan.

By nonanticipation and the chronological construction, the original and hybrid experiments have the same completed first-boundary law Q_1^δ : couple them with the same initial full book and pre-boundary innovations, and induct over the finite actual recursion. A future receiving prescription changes no earlier actual observation or action. Lemma 2, applied on each actual completed boundary of the original experiment, gives

$$\begin{aligned} J_j(u_j; \delta_j, \sigma_{-j}^*) &\leq \int \widehat{V}_j(h) Q_1^\delta(dh | u_j) \\ &= J_{j,1}(u_j; \delta_{1j}, \sigma_{1,-j}^*) \\ &\leq J_{j,1}(u_j; \sigma_{1j}^*, \sigma_{1,-j}^*) = J_j(u_j; \sigma_j^*, \sigma_{-j}^*). \end{aligned}$$

The middle inequality is the complete first-stage comparison just proved. Every expectation is a same-experiment actual-cash integral. In particular an assessment value at a later reference-null observation is never substituted for the posterior generated by the deviation.

At a start after first-stage completion, use the bookwise receiving envelope directly. At an already posted or committed receiving phase use its corresponding phase value in Lemma 2, without restarting the round. Original exits, receiving exits, automatic bookkeeping cuts and terminal states have their actual singleton cash continuation. These cases exhaust all starts. Each u_j is fixed separately; the assertion requires no intersection of uncountably many probability-one events. Lemmas 3 and 4 supply the same generated law, supported kernels, whole-own information and consistency. This establishes the claimed equilibrium.

It remains to identify its actual financing event. On each of the four paths in (9), the first round settles at zero with

$$k_z = B_0 \mathbf{1}_{\{z=+\}}, \quad x_1 = x_2 = (R_0 - k_z)/2 > 0.$$

Its cap table is

$$s_M^z = a(1 - R_0/P), \quad s_I^z = b(1 - R_0/P) + k_z/P, \quad s_{O_1}^z = s_{O_2}^z = (R_0 - k_z)/(2P).$$

These sum to one; actual capital is $k_z + x_1 + x_2 = R_0 > 0$. At T , first dividends are $s_j^z Y_\theta$ and shares survive. At t_2 , F settles its own $K > 0$; the new table is $((1 - r)s^z, r)$. At t_3 the old owners receive $s_j^z S$ and F receives K . Final cash is

$$\begin{aligned} C_M^{\text{fin}} &= s_M^z V_\theta, & C_I^{\text{fin}} &= B_0 - k_z + s_I^z V_\theta, \\ C_{O_i}^{\text{fin}} &= W_i - x_i + s_{O_i}^z V_\theta, & C_F^{\text{fin}} &= K. \end{aligned}$$

Both financing entries carry the same immutable ι , and $0 < T + \Delta$. Hence

$$\mathbb{P}^*\{\text{positive settled capital at } 0 \text{ and } T + \Delta \text{ for the same company}\} = \sum_{\theta, z} \pi_0(\theta, z) = 1.$$

This is unconditional, and it does not discard any feasible exit history from the assessment. The theorem is proved. $\square$

7 An exact witness and its economic interpretation

The domain is nonempty. Choose

$$\begin{aligned} I_0 = 1, \quad c = \frac{1}{5}, \quad B_0 = \frac{3}{10}, \quad a = \frac{7}{10}, \quad b = \frac{3}{10}, \quad q = \frac{1}{2}, \quad \eta = \frac{3}{4}, \quad Y_L = 2, \quad Y_H = 4, \\ T = \kappa = \bar{u} = 1, \quad J = N = 2, \quad (t_{br,1}, t_{br,2}) = (\frac{1}{3}, \frac{2}{3}), \\ (\ell_L^+, \ell_L^-) = (1, 2), \quad (\ell_H^+, \ell_H^-) = (2, 1), \\ W_1 = W_2 = K = 1, \quad Z = 2, \quad \Delta = 1. \end{aligned}$$

Every restriction in (1) is satisfied. Then $S = 1$, $(V_L, V_H) = (3, 5)$, $R_0 = 4/5$, $(\mu_+, \mu_-) = (9/2, 7/2)$, and

$$P = \frac{101}{26}, \quad \alpha_p = \frac{104}{505}, \quad r = \frac{1}{2}.$$

The first tables, in order (M, I, O_1, O_2) , are

$$\begin{aligned} s^+ &= (2807/5050, 1593/5050, 13/202, 13/202), \\ s^- &= (2807/5050, 1203/5050, 52/505, 52/505). \end{aligned}$$

Each second table halves these old fractions and gives F one half. First actual allocations (I, O_1, O_2) are $(3/10, 1/4, 1/4)$ at $+$ and $(0, 2/5, 2/5)$ at $-$. Capital 1 arrives at date 2, between first payout at 1 and final payout at 3.

The economic link is exact: future surplus $Z - K$ belongs pro rata to the surviving first cap table and enters current valuations V_θ . Accordingly future investment affects current dilution and the incumbent's signal-dependent capital supply. The construction has an immediate pooling path, a deterministic receiving opportunity, and an endowed receiving financier; testing, bridges and exit are available but unused on that path. The result establishes existence for this institution, while retaining complete deviation comparisons and every feasible history.

Source and version note

This exposition reproduces the mathematical content of *Two primary financings of one surviving company*, assembly version 2, and the supporting derivations T1.1–T1.8 version 1 in the S8–T1 proof package, dated September 2026. The proof above includes the accounting, exact domain, formal-round inequalities, complete timing/bridge bounds, full-book kernels, dynamic law, receiving envelope and cross-stage join needed for the stated theorem. No external or inaccessible proof reference is required to complete it. Programme authorship is Elies Lucio and Hugues Pouget, as credited in the integrated research manuscript. This presentation extract leaves the canonical research sources unchanged.