MANUSCRIPT EXTRACT

Fundraising over arbitrary state spaces

T2 technical supplement

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Section 4.3 and Section 5; original manuscript pages 19-21.

The existing same-safe-rate counterexample is followed by the general-state attainment theorem and an exact continuous-state value function. Theorem 5.1 and Proposition 5.2 appear with their complete proofs.

With positive integrable state prices, bounded output and finite opportunity capacity, the stated high-effort-or-saving contracting programme attains an optimum on an arbitrary probability space. Wealth value remains continuous and concave. Finite piecewise affinity need not survive.

This remains a conditional-price contracting result with the specified success compensation, contribution and saving choices. It does not derive endogenous opportunity capacity or a general equilibrium; effort comparisons have their stated additional conditions.

How to read this supplement

This is a manuscript extract, not a self-contained proof note. The mathematical text, theorem numbers and printed cross-references are retained. References to sections and bibliography entries outside the extract point to the complete manuscript. Boundary-page content outside the selected sections is visually trimmed. The relevant theorem statements and their full local proofs are retained.

4.3 A proved failure of scalar compression

Proposition 4.5 (Same safe return, different fundraising)**.** *Let two future states be equally likely. Set* $p = 4/5$*,* $p_0 = 2/5$*,* $Y = 3$*,* $b = 2/5$*,* $W = 1/10$ *and* $\bar{D} = 1$*. With* $m = (1/2, 1/2)$ *the optimum has* $(D, J) = (1/2, 2/5)$*. With* $m = (3/5, 2/5)$ *it has* $(D, J) = (5/6, 2/3)$*. Both state-price vectors imply the same safe gross return* $R = 2$*, and both satisfy the global low-effort bound.*

Proof. For constant prices, $A = 12/5$ and $P = 8/5$; Corollary 4.4 gives the first pair. For the stochastic prices, IC requires $z_L + z_H \geq 2D$, and LP participation is

$$\frac{6}{25}z_L + \frac{4}{25}z_H \leq \frac{1}{5}D + w.$$

Its left side is at least $8D/25$, so $3D/25 \leq w \leq 1/10$ and $D \leq 5/6$. Multiplying that participation inequality by $5/2$ also gives

$$\frac{2}{5}(z_L + z_H) - 2w \leq \frac{1}{2}D + \frac{1}{2}w - \frac{1}{5}z_L \leq \frac{7}{15}.$$

Choose $D = 5/6$, $w = 1/10$, $z_L = 0$, $z_H = 5/3$. Every constraint holds and the last bound is attained. Adding saved-wealth baseline $RW = 1/5$ gives $J = 2/3$. Finally, $A_0 + b = 8/5$ is below both 2 and $5/3$, the respective reciprocal maximum discount factors. □

The effect comes from shifting the manager's incentive compensation into the cheaper state. The safe rate cannot record that substitution. This is a conditional comparison of state prices, not yet a pair of fully reoptimized general equilibria. Its role in the integrated proof is precise: any macro interface passing only the safe rate discards a variable that the actual successor LP program uses.

5 Fundraising over arbitrary aggregate state spaces

Finite states are needed for the finitely many affine pieces in Proposition 4.1. The central attainment and wealth properties extend to arbitrary probability spaces. A continuous-state example proves that retaining the finite-piece conclusion would be false. These results concern the stated successor contract domain; they do not replace the separately solved nonlinear search-career program.

Theorem 5.1 (General-state successor fundraising)**.** *Let* $(S, \mathcal{F}, \mathbb{P})$ *be a probability space. Suppose* $m > 0$ *almost surely,* $m \in L^1$*,* $Y \in L^\infty_+$*, and measurable effort probabilities satisfy* $0 \leq p_0 < p \leq 1$

almost surely. Let $0 \le \bar{D} < \infty$, $b \ge 0$, *and* $R = (\mathbb{E}m)^{-1}$. *For liquid wealth* $W \ge 0$, *maximize*

$$G(W) = \max\{\mathbb{E}[pz] - Rw\}$$

over $D, w \in \mathbb{R}$ *and* $z \in L^\infty$, *subject to*

$$0 \le D \le \bar{D}, \quad 0 \le w \le \min(W, D), \qquad 0 \le z \le YD \quad \textit{almost surely},$$
$$\mathbb{E}[(p - p_0)z] \ge bD, \qquad \mathbb{E}[mp(YD - z)] \ge D - w.$$

The optimum is attained. Its high-effort-or-saving value $J(W) = RW + G(W)$ *is continuous and concave on* $[0, \infty)$. *The franchise* G *is nonnegative, nondecreasing and concave,* $J(W + h) \ge J(W) + Rh$, *and larger opportunity capacity weakly raises* G. *These conclusions require neither finitely many states nor a finite-support approximation. Finite piecewise affinity is not asserted.*

Proof. The zero contract is feasible. For W in any bounded interval, D, w lie in fixed compact intervals and $\|z\|_\infty \le \|Y\|_\infty \bar{D}$. The latter ball is compact for the weak-star topology $\sigma(L^\infty, L^1)$; the probability measure identifies L^∞ with the dual of L^1 for this purpose. This is the dual-ball compactness theorem; see, for example, Ramesh (2020, Theorem4.1).

The order constraints are closed in the product topology. Indeed, $0 \le z \le YD$ is equivalent to

$$\mathbb{E}[az] \ge 0, \qquad \mathbb{E}[a(YD - z)] \ge 0 \quad \text{for every } a \in L^1_+.$$

Each displayed functional is continuous. The remaining source and incentive constraints are closed because $p, p - p_0, mp \in L^1$ and mpY is integrable. The objective is continuous for the same reason. Consequently the feasible set is compact and the optimum is attained.

Convexly combine contracts at two wealth levels. All constraints, including the almost-sure order bounds, are preserved at the combined wealth. This proves concavity; feasible-set inclusion proves wealth and capacity monotonicity. The zero contract gives nonnegativity, and adding the linear saving value gives the claimed lower increment for J. The bounded output makes each value finite, so concavity gives continuity at every positive wealth.

For right continuity at zero, let $W_n \downarrow 0$. The nondecreasing G has a finite limiting value along this sequence. Corresponding optimizers lie in one compact product set. Take a convergent *subnet*: weak-star compactness on an arbitrary probability space does not imply sequential compactness. Its limiting contract satisfies $w = 0$ and every other zero-wealth constraint. Objective continuity identifies its value with the preceding limit, which is at most $G(0)$. Monotonicity gives the reverse inequality. Thus G, and hence J, is right-continuous. □

The statewise effort-upgrade argument also extends by integration. When

$$b \le \mathbb{E}\left[\frac{(p - p_0)^2}{p} Y\right],$$

replace any low-effort compensation by $z' = (1 - p_0/p)YD + (p_0/p)z$. Its limited liability, actual source, incentive inequality and payoff comparison hold almost surely and after integration exactly as in Proposition 4.3. All terms are integrable under the stated assumptions. This does not supply unrestricted contingent financial contracts or endogenous capacity.

Proposition 5.2 (An attained strictly curved fundraising value)**.** *Let s be uniform on $[0,1]$, and set*

$$m(s) = \frac{1+s}{3}, \qquad R = 2, \qquad Y = 3, \quad p = 1, \quad p_0 = 0, \quad b = 1, \quad \bar{D} = 1.$$

The globally optimal value over high effort, low effort and saving is

$$J(W) = \begin{cases} 3\sqrt{2+2W} - 3, & 0 \le W \le 1/8, \\ 2W + 5/4, & W \ge 1/8. \end{cases} \tag{15}$$

An optimizer has $D = 1$, $w = \min(W, 1/8)$ and

$$z(s) = 3\,\mathbf{1}\{s \le \sqrt{2+2w} - 1\}.$$

In particular J is strictly curved on $(0, 1/8)$ and is not a finite piecewise-affine function.

Proof. Fix $D > 0$ and $0 \le w \le D$. LP participation permits $\mathbb{E}[mz] \le D/2 + w$. Ignore IC temporarily and let $v = w/D$. For a given priced compensation budget, success money is most cheaply allocated at the lowest $m(s)$. The maximizing bounded schedule is $z = 3D\mathbf{1}_{[0,a]}$, where

$$a = \sqrt{2+2v} - 1 \in [\sqrt{2} - 1, 1], \qquad D(a + a^2/2) = D/2 + w.$$

For completeness, if z_* is this schedule, then $[1 - m(s)/m(a)](z(s) - z_*(s)) \le 0$ almost surely. Integrate and use the old compensation budget to obtain $\mathbb{E}z \le \mathbb{E}z_*$. This proves the global bound without assuming a smooth compensation optimizer.

Its incentive inequality is satisfied strictly at positive scale: $\mathbb{E}z_* = 3Da \ge 3D(\sqrt{2} - 1) > D = bD$. Thus the IC-free upper bound is actually feasible. The franchise value at fixed D, w equals $D\phi(w/D)$, where

$$\phi(v) = 3\sqrt{2+2v} - 3 - 2v.$$

At fixed w, its derivative with respect to D is

$$3\left[\frac{2+v}{\sqrt{2+2v}} - 1\right] > 0, \qquad 0 \le v \le 1,$$

since $(2+v)^2 - (2+2v) = v^2 + 2v + 2 > 0$. Increasing scale preserves $w \le D$, so every positive optimal contract has $D = 1$; at zero scale the positive zero-wealth franchise is already a strict improvement. Finally $\phi'(v) = 3/\sqrt{2+2v} - 2$ vanishes only at $v = 1/8$ and changes from positive to negative. Hence $w = \min(W, 1/8)$ and adding actual saved wealth yields (15).

The LP's actual expected priced residual is $D - w$, so its source is fully covered. Under low effort there is no output; participation requires $w = D$. The GP then receives at most $2(W - D) + D = 2W - D \le 2W$. Thus low effort cannot improve the displayed high-effort-or-saving optimum. For $0 < W < 1/8$, $J''(W) = -3(2+2W)^{-3/2} < 0$, proving failure of finite piecewise affinity. □